ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, No. 2, 1976

ALTERNATING HAMILTONIAN CYCLES

BY

BÉLA BOLLOBÁS AND PAUL ERDÖS

ABSTRACT

Colour the edges of a complete graph with n vertices in such a way that no vertex is on more than k edges of the same colour. We prove that for every k there is a constant c_k such that if $n > c_k$ then there is a Hamiltonian cycle with adjacent edges having different colours. We prove a number of other results in the same vein and mention some unsolved problems.

Given the natural numbers n and d, denote by K_n $(\Delta_c \leqq d)$ a *complete graph* with n vertices whose *edges* are coloured in such a way that *no vertex is on more than d edges of the same colour.* [We denote by Δ_c the maximal degree in the subgraph formed by the edges of colour c.] These graphs were examined by Daykin [1], who proved that if $d = 2$ and $n \geqq 6$ then every such graph contains a *Hamiltonian cycle* whose adjacent edges have different colours. Daykin [1] also asked whether this holds for every d and every sufficiently large n (depending on d). We shall answer this question in the affirmative. We shall also prove a number of related results; among others we shall give partial solutions to other problems stated in [1].

Denote by AC_l a cycle of length l in which adjacent edges have different colours. These are the *alternating cycles* and the *alternating paths* are defined analogously. Our main result about the existence of an AC_n in a K_n $(\Delta_c \leqq d)$ will be proved by using certain auxiliary subgraphs, subgraphs in which it is particularly easy to construct alternating paths. Let us show first that K_n $(\Delta_c \leqq d)$ contains a large subgraph with a stricter condition on the degree.

LEMMA 1. *Let $n \geqq d \geqq \delta \geqq 1$ be natural numbers and let r be a natural number such that*

$$r^{1+2/\delta} d < n.$$

Received January 10, 1975

Then every $G = K_n\,(\Delta_c \leqq d)$ contains an $H = K_r\,(\Delta_c \leqq \delta)$. In particular, if $64d < n$ then every $G = K_n\,(\Delta_c \leqq d)$ contains a $K_4(\Delta_c \leqq 1)$.

PROOF. Denote by $\mathscr{A}$ the set of complete subgraphs of G with r vertices and if x is a vertex of G, let

$$\mathscr{A}_x = \{L \in \mathscr{A} : L \text{ contains at least } \delta + 1 \text{ edges of the same colour, ending at } x\}.$$

Denote by $d_1, \cdots, d_l$ the degrees of x in the subgraphs formed by the various colour classes. Then $d_i \leqq d$ and $\Sigma_1^l\, d_i = n - 1$, so by the convexity of $f(t) = \binom{t}{u}$ we have

$$|\mathscr{A}_x| \leqq \sum_{i=1}^{l} \binom{d_i}{\delta+1}\binom{n-(\delta+2)}{r-(\delta+2)} \leqq \frac{n-1}{d}\binom{d}{\delta+1}\binom{n-(\delta+2)}{r-(\delta+2)}.$$

Consequently, if $\mathscr{B} = \cup\, \mathscr{A}_x$, where the union is over all vertices,

$$|\mathscr{B}|/|\mathscr{A}| = |\mathscr{B}| \Big/ \binom{n}{r} \leqq \frac{n(n-1)}{d}\binom{d}{\delta+1}\binom{n-(\delta+2)}{r-(\delta+2)} \Big/ \binom{n}{r} < n^{-\delta} r^{\delta+2} d^{\delta} < 1.$$

Thus $|\mathscr{A}| > |\mathscr{B}|$ and by construction every $H \in \mathscr{A} - \mathscr{B}$ will do for the lemma.

Denote by $V(G)$ the vertex set of a graph G. If $a, b \in V(G)$, $c(ab)$ denotes the colour of the edge ab.

LEMMA 2. *Suppose $G = K_n\,(\Delta_c \leqq d)$ contains an alternating path P that ends at a vertex x, a vertex y not on P and $s \geqq d/4$ vertex disjoint $K_4(\Delta_c \leqq 1)$ subgraphs, say $H_1, H_2, \cdots, H_s$. Then the following assertions hold.*

(i) *There is an index l such that P can be continued to an alternating path P^* that goes through the four vertices of H_l.*

(ii) *If $a \in V(H_1)$, $b \in V(H_2)$ and i, j are given colours, there is an alternating path Q from a to b, going through the eight vertices of H_1 and H_2 such that the first edge of Q does not have colour i and the last edge does not have colour j.*

PROOF. (i) Denote by k the colour of the last edge of P. As there are at most $d - 1$ other edges of colour k ending at x, there is an index l such that at least one of the edges joining x to H_l has colour different from k. Let k_1, k_2, k_3, k_4 be the vertices of H_l. We can suppose without loss of generality that $c(xh_1) \neq k$ and $c(h_3h_4) \neq c(h_4y)$. Then we can put $P^* = Ph_1h_2h_3h_4y$.

(ii) Denote the vertices of H_1 by $a = a_1, a_2, a_3, a_4$ and the vertices of H_2 by $b = b_1, b_2, b_3$ and b_4. We can suppose without loss of generality that

$c(a_1 a_2) \neq i \neq c(a_1 a_3)$ and $c(b_1 b_2) \neq j \neq c(b_1 b_3)$. Furthermore, as $c(a_2 a_4) \neq c(a_3 a_4)$ and $c(b_2 b_4) \neq c(b_3 b_4)$, by symmetry we can suppose that $c(a_3 a_4) \neq c(a_4 b_4) \neq c(b_3 b_4)$. Then we can put $Q = a_1 a_2 a_3 a_4 b_4 b_3 b_2 b_1$.

Our first main result is an almost immediate consequence of these lemmas.

THEOREM 1. *If $69d < n$ then every $G = K_n(\Delta_c \leqq d)$ contains an alternating Hamiltonian cycle.*

PROOF. As $n - 4[5d/4] > 64d$, by Lemma 1 the graph G contains $s = [5d/4] + 1$ vertex disjoint $K_4(\Delta_c \leqq 1)$ subgraphs, say $H_1, H_2, \cdots, H_s$. Let P be a maximal alternating path in $H = G - \bigcup_1^s H_i$. Then $H - P$ contains at most $d - 1$ vertices. By Lemma 2(i) in G the path P can be continued to an alternating path P^* containing all these vertices and the vertices of at most $d - 1$ of the graphs H_i. Consequently there are $t \geqq [d/4] + 2$ subgraphs disjoint from P^*, say $H_1, H_2, \cdots, H_t$. Denote by x_1 (resp. x_2) the first (resp. last) vertex of P^* and by i_1 (resp. i_2) the colour of the first (resp. last) edge of P^*. There are at most $(d-1)/4$ subgraphs H_i such that every edge joining x (resp. y) to a vertex of H_i has colour i_1 (resp. i_2). Therefore one can find vertices $a_1 \in V(H_i), a_2 \in V(h_j)$, $1 \leqq i \neq j \leqq t$, such that $c(x_1 a_1) \neq i_1$ and $c(x_2 a_2) \neq i_2$.

By Lemma 2(ii) there is an alternating path Q from a_2 to a_i going through all the vertices of $\bigcup_1^t H_i$ such that the colour of the first edge is not $c(x_2 a_2)$ and the colour of the last edge is not $c(x_1 a_1)$. Then $a_1 x_1 P^* x_2 a_2 Q a_1$ is clearly an alternating Hamiltonian cycle. This completes the proof of the theorem.

REMARKS. 1. Exactly the same proof gives that under the conditions given in the theorem every $G = K_n(\Delta_c \leqq d)$ contains an AC_l for every $l, 3 \leqq l \leqq n$.

2. In the first version of the paper we proved Theorem 1 under the condition $n > c_\varepsilon d^{2+\varepsilon}$ ($\varepsilon > 0$), and only the referee's remarks made us prove this stronger form. A similar result has been proved independently by Chen and Daykin. Though the bound $n > 69d$ might not seem to be too bad, we suspect that it is very far from being the best possible, since from below we can construct practically nothing. (See the first conjecture at the end of the paper.)

Let us examine now the related questions. These questions arose in connection with the auxiliary subgraphs used in the proof of Theorem 1, but we think they are interesting on their own. Let $\alpha > 0$ be a given *constant.* How large does n have to be if every $K_n(\Delta_c \leqq d)$ contains a $K_{\alpha d}(\Delta_c \leqq 1)$? How large does n have to be if every $K_n(\Delta_c \leqq d)$ contains a complete subgraph with at least αd vertices without 2 edges of the same colour? We cannot give a complete answer to either of these questions but we prove reasonably good estimates.

THEOREM 2. a) *If* $\alpha^3 d^4 < n$ *then every* K_n $(\Delta_c \leqq d)$ *contains a* $K_{\alpha d}$ $(\Delta_c \leqq 1)$.

b) *There is a constant* C *such that if* $d^3 > Cn(\log n)^3$ *then there is a* K_n $(\Delta_c \leqq d)$ *that does not contain a* $K_{[\alpha d]}(\Delta_c \leqq 1)$.

PROOF. The first part is contained in Lemma 1. To prove b) we apply a probabilistic argument.

It will be clear from the argument that it is sufficient to prove the result when $k = n^{1/3}$ is an integer and n is sufficiently large.

Colour the edges of a K_n (complete graph with n vertices) with n/k colours, giving each colour probability k/n. Then with probability $>1/2$ the obtained graph G will be a K_n $(\Delta_c \leqq d)$, where $d = [k \log n]$. Let us choose a complete subgraph H of G with $r+1 = [\alpha d]$ vertices. If x is a vertex of H, the probability that H does not contain 2 edges ending at x that have the same given colour (say colour 1) is

$$\left(1-\frac{k}{n}\right)^r + r\left(1-\frac{k}{n}\right)^{r-1}\frac{k}{n} < 1-\frac{r^2k^2}{2n^2}.$$

Consequently the probability that H does not contain 2 adjacent edges of the same colour is at most

$$\left(1-\frac{r^2k^2}{8n_2}\right)^{m/(2k)}.$$

Now

$$\left(1-\frac{r^2k^2}{8n^2}\right)^{m/(2k)}\binom{n}{r} < 2\exp(-kr^3/(16n))\binom{n}{r} \to 0$$

as $n \to \infty$. In particular, if n is sufficiently large, the probability that H is a K_{r+1} $(\Delta_c \leqq 1)$ is $< \binom{n}{r}^{-1}/2$. Thus there exists a $G = K_n$ $(\Delta_c \leqq d)$ that does not contain a $K_{[\alpha d]}(\Delta_c \leqq 1)$, as claimed.

THEOREM 3. a) *If* $r^4 d < n$ *then every* K_n $(\Delta_c \leqq d)$ *contains a complete subgraph with* r *vertices without* 2 *edges of the same colour.*

b) *There is a constant* C *such that if* $d^4 > Cn(\log n)^4$ *then there is a* K_n $(\Delta_c \leqq d)$ *in which every complete subgraph with* $r = [\alpha d]$ *vertices contains* 2 *edges of the same colour.*

PROOF. The proof of the first part is analogous to the proof of Lemma 1 and the proof of the second part is exactly the same probabilistic argument as the proof of Theorem 2b). We omit the details.

Let us denote by K_n $(\chi_v \geqq \lambda)$ a *complete graph* with n vertices whose *edges* are coloured in such a way that *each vertex is on at least* λ *edges of different colour.* [χ_v = number of colours appearing among the edges containing a vertex v.] Daykin posed the question of finding a λ, as small as possible, such that every K_n $(\chi_v \geqq \lambda)$ contains an alternating Hamiltonian cycle. We shall show that $\lambda \geqq (7/8)n$ will do. We also give an example showing that $\lambda = [(n+2)/3]$ will no longer do.

THEOREM 4. *Every* $K_n\,(\chi_v \geqq (7/8)n)$ *contains an alternating Hamiltonian cycle.*

PROOF. Put $\varepsilon = 1/8$ and let $G = K_n\,(\chi_v \geqq (1-\varepsilon)n)$. If $e = xy$ is an edge of G, let $c(e) = c(xy)$ be its colour. Call an edge xy of G *x-unique* if $c(xy) \neq c(xz)$ if $z \neq y$. Call an edge xy *unique* if it is both x-unique and y-unique.

Let C be a cycle of maximal length in G, say length l, consisting of *unique* edges. As there are at least $(1-2\varepsilon)n$ x-unique edges for each vertex x, there are at least $(1-2\varepsilon)n^2 - \binom{n}{2} = (\frac{1}{2} - 2\varepsilon)n^2 + (n/2) = (n^2/4) + (n/2)$ *unique* edges. Therefore $l \geqq n/2$ (see [2]).

Let L_1 be a longest alternating path in $G - C$, let L_2 be a longest alternating path in G–C–L_1, etc. Suppose we obtain the paths $L_1, L_2, \cdots, L_t$ with $l_1, l_2, \cdots, l_t$ *vertices*, respectively. Then $l + \Sigma_1^t\, l_i = n$ and $l_i \geqq 2$ if $i < t$.

It is easily seen that if L_t is an $a_t b_t$ -path, where a_t might coincide with b_t, then C contains adjacent vertices c_t, d_t such that the path $c_t a_t L_t b_t d_t$ is an *alternating* path. Suppose now that L_s $(s > t)$ in an $a_s b_s$ -path, beginning with the edge $a_s a'_s$ and ending with the edge $b'_s b_s$. Then at most $\varepsilon n - 2 - \Sigma_{s+1}^t\, l_i$ of the edges $a_s c$, c a vertex of C, have the same colour as $a_s a'_s$, and a similar assertion holds for b_s. It is easily checked that

$$2\left(\varepsilon n - 2 - \sum_{s+1}^{t} l_i\right) + 2(t-s) + 1 < l.$$

Therefore one can choose inductively *different* vertices of C, say $c_t, d_t, c_{t-1}, d_{t-1}, \cdots, c_1, d_1$, such that c_i and d_i are *adjacent vertices* of C and the *paths* $P_i = c_i a_i L_i b_i d_i$ *are alternating,* $i = t, t-1, \cdots, 1$. Replacing the edge $c_i d_i$ of C by the path P_i, we obtain an alternating Hamiltonian cycle, as required.

OPEN PROBLEMS AND CONJECTURES. It is likely that Theorems 1 and 4 (our main results) can be strengthened considerably. The values at which these theorems are known to fail are much smaller than the bounds we needed to prove the existence of alternating cycles and we suspect that these rather feeble looking examples are nearer to the truth than our positive results.

1. Let $n = 4k + 1$. Then the edges of K_n can be coloured with red and blue in such a way that at each vertex there are $2k$ red and $2k$ blue edges. This is a $K_{4k+1}(\Delta_c \leqq 2k)$ that does not contain an AC_{4k+1}. We do not know a K_n $(\Delta_c \leqq d)$ with $d < [n/2]$ that does not contain an AC_{4k+1} and we suspect that there might not be one. So the problem is the following. *Does every* $K_n(\Delta_c \leqq [n/2]-1)$ *contain an alternating Hamiltonian cycle?*

2. Let $k = [(n-1)/3]$ and colour the edges of K_n with colours $0, 1, \cdots, k+1$ in the following way. Let $x_0, x_1, \cdots, x_{k-1}$ be k arbitrary vertices of K_n and divide the remaining vertices into k non-empty classes, $S_0, S_1, \cdots, S_{k-1}$. If $y \in S_j$ colour the edge $x_i y$ with the colour $|i-j|$. Use the colour k to colour the edges $x_i x_j$ and the edges yz, $y, z \in \bigcup_0^{k-1} S_i$. In this colouring of K_n with $k+1$ colours *every vertex is on an edge of each colour.* Clearly there is no alternating Hamiltonian cycle since every Hamiltonian cycle has three consecutive vertices in $\bigcup_0^{k-1} S_i$. It is not impossible that this example is essentially best possible, perhaps even without the restriction that each vertex is on an edge of each colour. In other words can Theorem 4 be sharpened to the following?

Every K_n $(\chi_v \geqq [(n+5)/3])$ *contains an alternating Hamiltonian cycle.*

References

1. D. E. Daykin, *Graphs with cycles having adjacent lines different colours*, to appear.

2. P. Erdös and T. Gallai, *On maximal paths and circuits of graphs*, Acta Math. Acad. Sci. Hungar. **10**, (1959), 337-356.

Department of Pure Mathematics
University of Cambridge
Cambridge, England